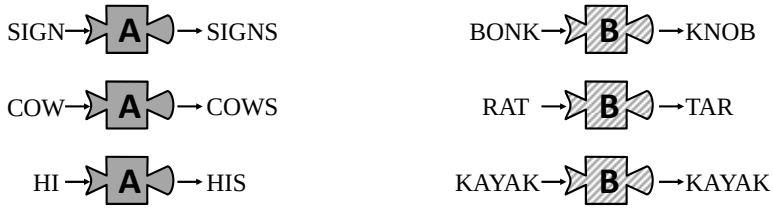
CalcActivity 1.3: Compositions of Functions

(How do you evaluate the composition of two functions?)

Model 1: Word Machines

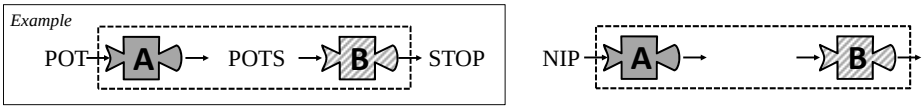


Construct Your Understanding Questions (to do in class)

1. Describe the effect of a) Machine A on a word b) Machine B on a word.

2. Consider a combination of Machines A and B such that the output of A is used as the input for B. See Example below, left. This is called a **composition** of these machines.

   a. Perform the same series of operations using "NIP" as the input (below, right).



   b. By convention, the **composition** contained within the dotted box above is called " $B \circ A$ " (which is read "B composed with A"). Label both dotted boxes above with the caption **Machine** $B \circ A$.

   c. For the composition in part a., which machine appears in the name first **A** or **B** [circle one]?

   d. Which machine is evaluated first **A** or **B** [circle one]?

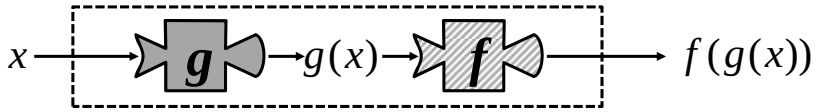
3. Using the naming convention from above, construct a name for this **composition** (shown two times below) and label each dotted box below with this name.

   a. Fill in the missing outputs of this machine.

   b. Does Machine $A \circ B$ (in this question) produce a different output from Machine $B \circ A$ (found in the previous question)? In other words, is the order of operation important?

Model 2: Composition of Functions

It is possible to **compose** functions as we composed machines on the previous page. The result, a composite function, can be written as shown below, or using the symbol $\circ$.



Composition of Two Functions Composite Function

Construct Your Understanding Questions (to do in class)

4. Based on the naming conventions from the previous page, which is the correct name of the composite function in Model 2? $f \circ g$ or $g \circ f$ [circle one and explain your reasoning]

5. Assume $f(x) = x^2$ and $g(x) = x + 1$

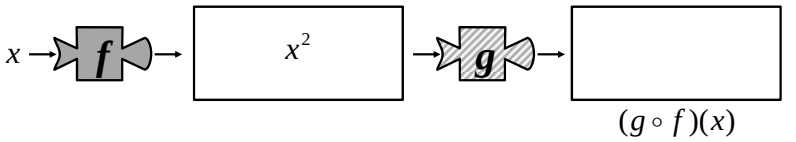
   a. What is the output of $g \circ f$ with an input of $x = 3$?

   This can be written $(g \circ f)(3) =$

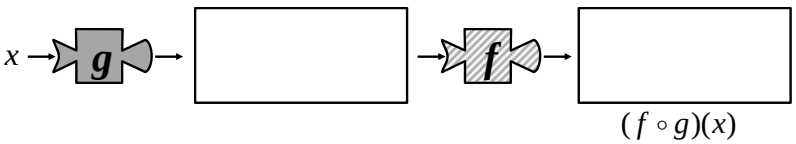
   b. What is the output of $f \circ g$ with an input of $x = 3$?

   This can be written $(f \circ g)(3) =$

   c. You can arrive at an algebraic expression for a composition by using x as the input. Fill in each of the following boxes to determine $(g \circ f)(x)$, and then $(f \circ g)(x)$. The first box is filled in for you.



$(g \circ f)(x)$



$(f \circ g)(x)$

   d. For f and g as defined in this question, does $f \circ g = g \circ f$? Explain.

6. (Check your work) Substitute $x = 3$ into the equation you derived…

   a. for $(g \circ f)(x)$ and check that your answer is 10.

   b. for $(f \circ g)(x)$ and check that your answer is 16.

   Use this to also check your answers to parts a) and b) of Question 5. If any of your group's answers do not fit, check your methods against those of another group.

7. The function $f(x) = \sqrt{x+3}$ can be expressed as a composition of two simpler functions, *h* and *g* where $h(x) = \sqrt{x}$ and $g(x) = x + 3$.

   For these functions *f, g,* and *h,* does $f(x) = (g \circ h)(x)$ <u>or</u> $f(x) = (h \circ g)(x)$ [circle one]?

8. Consider the function $f(x) = (x-1)^2$. Propose two functions *g* and *h* such that $(g \circ h)(x) = f(x) = (x-1)^2$.

9. Later in this course we will discover that it is very useful to ***de*compose** a function into simpler functions (as you were asked to do in the previous two questions). For the following list of functions, propose two functions $g(x)$ and $h(x)$ such that $f(x) = (g \circ h)(x)$**.** You may <u>not</u> use the functions $f(x) = x$ or $g(x) = x$. (Construct an explanation for why using the function x would make this question too easy.)

   a. $f(x) = (x+4)^3$

   b. $f(x) = \sqrt{x^2 + 7x}$

   c. $f(x) = \dfrac{\sqrt{x}}{2}$

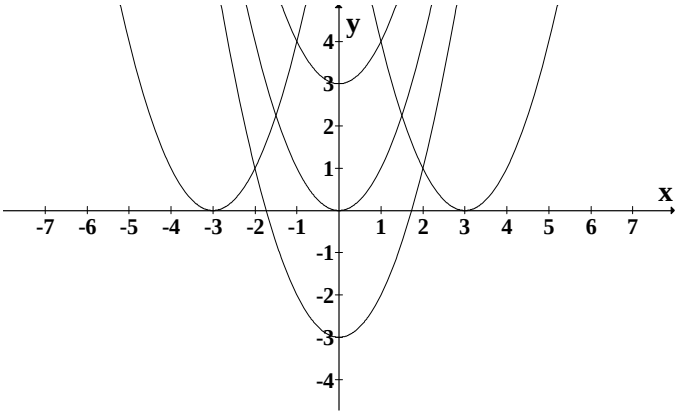
   d. $f(x) = \sqrt{\dfrac{x}{2}}$

   e. $f(x) = x^3 + 4$

   f. $f(x) = \dfrac{\sqrt{x-1}}{1+\sqrt{x-1}}$

Model 3: Shifting

At right is the parabola $h(x) = x^2$, along with four related parabolas, each one identical to $h(x) = x^2$, but shifted by a distance of 3 units up, down, right, or left, respectively.



Construct Your Understanding Questions (to do in class)

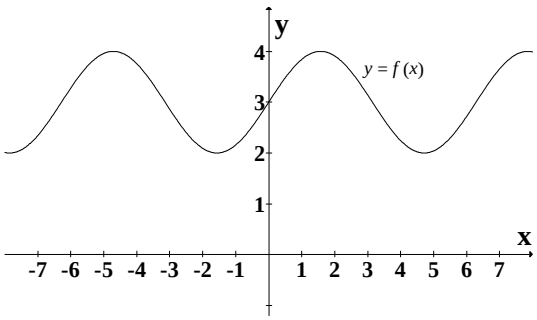
10. In Model 3, label the graph of $h(x) = x^2$ and the four other functions listed in the column labeled "**Function**" on the table below. Check your work by finding the vertex of each parabola and confirming that this point is satisfied by the equation you chose as a match.
    a. Complete Column 2 by filling in the word LEFT, RIGHT, UP, or DOWN.
    b. Complete Column 3: Each shift of $h(x) = x^2$ can be represented as a composition of the original function $h(x) = x^2$ and another function $g(x)$. For each row, propose a function $g(x)$ that accomplishes this shift. [The first one is done for you.]
    c. In Column 4 indicate if $h \circ g$ or $g \circ h$ give the function in Column 1.

Function	**Direction of Shift from** $h(x) = x^2$	***g*(*x*) that can accomplish this shift**	$h \circ g$ or $g \circ h$ **[indicate which] gives *f*(*x*) shown in Column 1**
$f(x) = x^2 + 3$		$g(x) = x + 3$	
$f(x) = x^2 - 3$			
$f(x) = (x + 3)^2$			
$f(x) = (x - 3)^2$			

11. (Check your work) $f(x)$ shown at right is the graph of, a sine wave shifted up by 3 units.

    Assume $h(x) = \sin x$

    Which function $g(x)$ from the previous question and which composition $h \circ g$ or $g \circ h$ or **both** [circle all that apply] will give the function $f(x)$ shown at right?



12. (Check your work) In general, if you want to shift a function $h(x)$ a <u>vertical</u> distance ***c*** (down or up) to produce a new shifted function $f(x)$, you can compose $h(x)$ with another function $g(x) = x + c$ to give $f(x)$.

    a. (Check your work) Check that this information is consistent with your answer to the previous question.

    b. c in the function $g(x) = x + c$ can be positive or negative. Describe the result of the composition when it is positive versus negative.

    c. Which composition $h \circ g$ <u>or</u> $g \circ h$ <u>or</u> **both** [circle all that apply] will give the function $f(x)$, as described above?

13. Suppose you want to shift a function $h(x)$ a <u>horizontal</u> distance ***c*** (left or right) to produce a new shifted function $f(x)$. Write a function $g(x)$ and an expression stating how $h(x)$ and $g(x)$ can be composed to give $f(x)$. [Hint: look back at the horizontal shifts in Question 10.]

14. Assume: $h(x) = x^2$ $g_v(x) = x + 1$ and $g_h(x) = x - 2$

    a. Describe the effect of the composition $g_v \circ h \circ g_h$ in terms of a) horizontal and b) vertical shifting of the graph of the parabola in comparison to the original function *h*.

    b. (Check your work) Is your answer to part a) consistent with the fact that this shifted parabola will have its vertex at the point (2,1)?

    c. Evaluate the composition $(g_v \circ h \circ g_h)(x)$ so as to generate the equation in terms of *x* for this shifted parabola. Remember to evaluate the functions in the name from right to left (i.e., evaluate g_h first).

15. (Check your work) Is your answer to the previous question consistent with Summary Box 1.3.1?

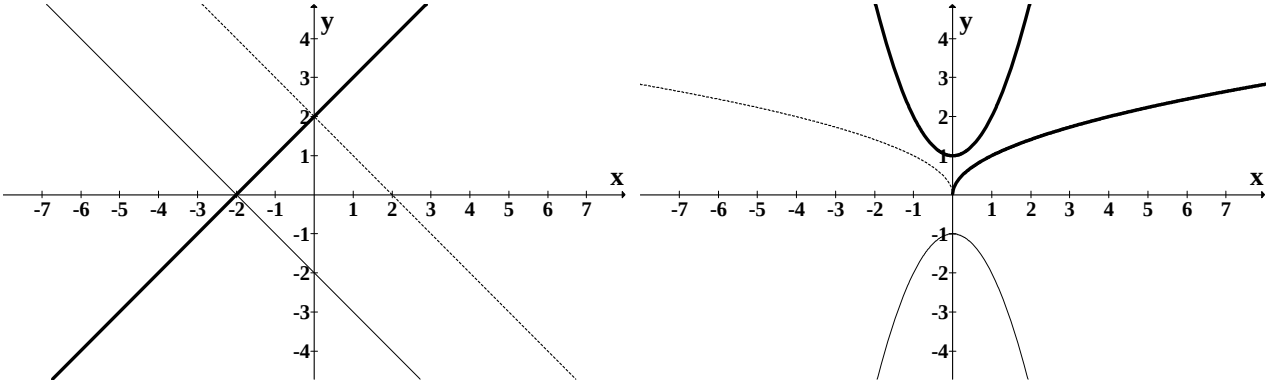
Summary Box 1.3.1: Shifting as Described by Compositions of Functions

When $g_h(x) = x + c_h$ and $g_v(x) = x + c_v$ the composition $g_v \circ h \circ g_h$ shifts the function $h(x)$…

- in the horizontal direction by an amount c_h
- in the vertical direction by an amount c_v

16. In Summary Box 1.3.1…
    a. when c_h is positive, the horizontal shift is **right** or **left** [circle one].
    b. when c_v is positive, the vertical shift is **up** or **down** [circle one].

Model 4: Reflecting About the *x* and *y* Axes



Extend Your Understanding Questions (to do in or out of class)

17. Identify the graph of the function $f(x) = x + 2$ in Model 4.
    a. Which line is a reflection of $f(x)$ about the *x* axis:
    The **light solid line** or the **dashed line** [circle one] in Model 4?
    b. Which line is a reflection of $f(x)$ about the *y* axis:
    The **light solid line** or the **dashed line** [circle one] in Model 4?
    c. Write an equation for each of these lines.

18. (Check your work) Are your answers on the previous page consistent with the following?

    The functions $f(x) = x + 2$ and $-f(x) = -x - 2$ are reflections of each other about the *x* axis.

    The functions $f(x) = x + 2$ and $f(-x) = -x + 2$ are reflections of each other about the *y* axis.

19. The functions $f(x) = x^2 + 1$ and $f(x) = \sqrt{x}$ are shown in **bold** on the right of Model 4.

    a. Identify the equations of the light solid curve, and the dotted curve on this graph.

    b. The reflection of $f(x) = \sqrt{x}$ about the *x* axis is not shown in Model 4. Add it and label it with its equation.

20. A function $f(x)$ can be composed with the function $g(x) = -x$ to give a reflection of $f(x)$.

    a. Which composition ($f \circ g$ or $g \circ f$) gives the reflection of $f(x)$ about the *x* axis? [circle one]

    b. Which composition ($f \circ g$ or $g \circ f$) gives the reflection of $f(x)$ about the *y* axis? [circle one]

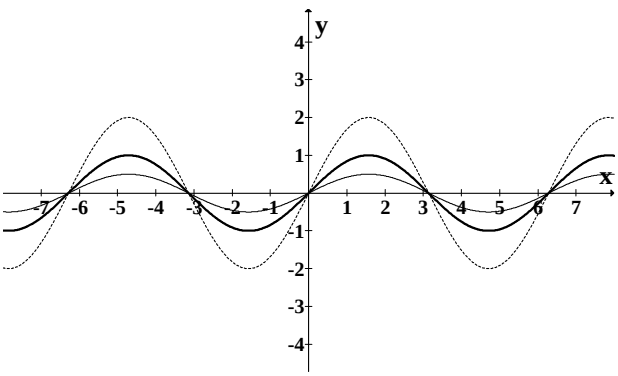
21. (Check your work) Check your conclusions above with at least one other group, then complete Summary Box 1.3.2 by filling in each blank with $g \circ f$ or $f \circ g$, as appropriate.

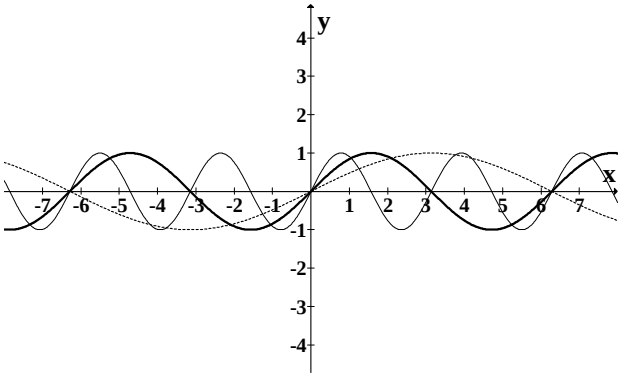
Summary Box 1.3.2: Reflecting of Functions

For a function $f(x)$, a composition with the function $g(x) = -x$ results in a reflection about the…

- *x* axis when the composition is _____
- *y* axis when the composition is _____

Model 5: Stretching and Compressing of Functions





Extend Your Understanding Questions (to do in or out of class)

22. Each graph in Model 5 shows the function $h(x) = \sin x$ in **bold**. Determine which graph in Model 5 shows $h(x)$ along with versions of $h(x)$ that are stretched and compressed in ….
    a. the vertical direction. (label this graph "vertical stretch/compress")
    b. the horizontal direction. (label this graph "horizontal stretch/compress")

23. Complete the table in Summary Box 1.3.3 by writing the correct pair of terms in each empty box. Choose from: **compressed vertically**, **compressed horizontally**, **stretched vertically**, **stretched horizontally**.

Summary Box 1.3.3: Stretching and Compressing of Functions

For a function $h(x)$, a composition with the function $g(x) = cx$ (where c is positive) results in stretching or compressing of the original function to give a new function $f(x)$.

	$0 > c > 1$ (e.g. ½)	$c > 1$ (e.g. 2)
$f = g \circ h$	Compared to $h(x)$, $f(x)$ is…	Compared to $h(x)$, $f(x)$ is…
$f = h \circ g$	Compared to $h(x)$, $f(x)$ is…	Compared to $h(x)$, $f(x)$ is…

Confirm Your Understanding Questions (to do at home)

24. Use a graphing program to plot the graph of the function $f(x) = ax^2$ where $a = 1$. Then also plot the graph of the function $f(x) = ax^2$ where the value of a is given below. You will have 4 graphs when you are finished. For each value of a, describe in words the effect of changing a from $a = 1$ to…
    a. $a = -1$
    b. $a = 3$
    c. $a = 10$

25. Use a graphing program to plot the function $f(x) = (x + c)^2$ where $c = 0$ and the changes noted below. For each change, describe in words the effect of changing c from $c = 0$ to…
    a. $c = -1$
    b. $c = 3$
    c. $c = 10$

26. Use a graphing program to plot the function $f(x) = x^2 + d$ where $d = 0$ and the changes noted below. For each change, describe in words the effect of changing d from $d = 0$ to…
    a. $d = -1$
    b. $d = 3$
    c. $d = 10$

27. Write the equation the parabola $f(x) = x^2$ (shown below) shifted a distance *c* in the direction indicated by each arrow.